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Optimum Measurements for Discrimination Among Symmetric Quantum States and Parameter Estimation

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An optimum quantum measurement that minimizes the average probability of error is considered for symmetric quantum states. The positive operator-valued measure (POM) which satisfies the necessary and sufficient condition for the minimization of the average probability of error is derived by using the quantum Bayes strategy. It is also shown that the mutual information obtained in the optimum quantum measurement becomes extremum. Furthermore, an optimum quantum measurement for parameter estimation is found for symmetric quantum states by applying the maximum-likelihood estimation. The optimum POM for the parameter estimation has the same structure as that for the quantum state discrimination.

1. INTRODUCTION

It is well known in quantum mechanics that nonorthogonal quantum states cannot be distinguished with certainty by means of any quantum measurement. This causes the degradation of detectability of quantum states in measurement processes and limits the efficiency of information transmission in quantum communication systems. Thus it is important in quantum measurement theory as well as quantum information theory to find an optimum quantum measurement that can distinguish among nonorthogonal quantum states as accurately as possible (Helstrom, 1969, 1976). Since coherent states used in optical communications with laser and fiber technology are nonorthog-

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onal, finding the optimum measurement is also useful for design of high-performance signal receivers. Quantum detection or decision theory is applied for investigating such problems (Helstrom, 1976). On the other hand, when a quantum state includes unknown parameters, it is important in quantum measurement theory to estimate the values of these parameters by performing some measurement on the quantum state (Helstrom, 1973, 1974; Holevo, 1978). Estimating the phase shift of a photon state induced by some optical device is one typical problem. Such a problem is solved by using quantum estimation theory (Helstrom, 1976; Holevo, 1982). To obtain optimum measurements in quantum detection and estimation theories, we have to solve nonlinear operator equations which are very complicated.

Therefore the purpose of this paper is to obtain optimum quantum measurements explicitly for discrimination among symmetric pure quantum states with equal *a priori* probabilities and parameter estimation for symmetric quantum states. In Section 2, the optimum quantum measurement that minimizes the average probability of error is derived by using the quantum Bayes strategy. It is shown that the mutual information obtained in this quantum measurement becomes extremum. In Section 3, applying the maximum-likelihood estimation, we obtain the optimum measurement for parameter estimation for symmetric quantum states. In Section 4, we summarize our results.

2. DISCRIMINATION AMONG SYMMETRIC QUANTUM STATES

2.1. Minimization of the Average Probability of Error

In this section, applying the quantum Bayes strategy, we obtain an optimum measurement for discrimination among symmetric quantum states. Suppose that a physical system takes one of M symmetric quantum states with equal probabilities. Then we would like to find an optimum quantum measurement whose outcome allows us to know as accurately as possible which quantum state is realized. The M symmetric quantum states which should be distinguished by means of the quantum measurement are described by statistical operators,

$$\begin{aligned}\hat{\rho}_j &= |\psi_j\rangle\langle\psi_j| \\ &= \hat{V}^{j-1}|\psi\rangle\langle\psi|\hat{V}^{\dagger j-1} \end{aligned} \tag{2.1}$$

where $j = 1, 2, \ldots, M$. In this paper, we do not assume that the state vectors $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_M\rangle$ in equation (2.1) are linearly independent. Thus our result is valid for both linearly independent and linearly dependent state vectors. In equation (2.1), the operator $\hat{V}$ is unitary ($\hat{V}\hat{V}^{\dagger} = \hat{V}^{\dagger}\hat{V} = \hat{I}$) and satisfies the relation

$$\hat{V}^{M} = \hat{I} \tag{2.2}$$

where $\hat{I}$ stands for an identity operator. This equation gives the relation of the statistical operator, $\hat{\rho}_{j\pm M} = \hat{\rho}_j$. Each quantum state is assumed to have equal *a priori* probability p_j, namely

$$p_j = 1/M \tag{2.3}$$

We perform a quantum measurement to know the quantum state $\hat{\rho}_j = |\psi_j\rangle\langle\psi_j|$ as accurately as possible. Such a quantum measurement can be described by a positive operator-valued measure (POM) (Helstrom, 1976; Holevo, 1978), denoted as $\hat{\Pi}_j$, which satisfies the following relations:

$$\hat{\Pi}_j \geq 0 \tag{2.4}$$

$$\sum_{j=1}^{M} \hat{\Pi}_j = \hat{I} \tag{2.5}$$

It should be noted that the POM which describes an optimum quantum measurement does not always become a projection operator.

Using the POM $\hat{\Pi}_j$ of the quantum measurement and the statistical operator $\hat{\rho}_j$ of the quantum state, we can express the conditional probability $P(j|k)$ as

$$P(j|k) = \mathrm{Tr}[\hat{\Pi}_j \hat{\rho}_k] \tag{2.6}$$

which means the probability that we infer the quantum state $\hat{\rho}_j$ when the quantum state $\hat{\rho}_k$ is true, where Tr stands for the trace operation. Then the quantity $P(j|k)$ ($j \neq k$) represents the detection-error probability and the average probability of error P_e is given by

$$\begin{aligned} P_e &= \frac{1}{M} \sum_{j=1}^{M} \sum_{k=1(\neq j)}^{M} P(j|k) \\ &= 1 - \frac{1}{M} \sum_{j=1}^{M} P(j|j) \end{aligned} \tag{2.7}$$

where we have used the relation $\Sigma_{j=1}^{M} P(j|k) = 1$. Therefore our task is to find the POM $\hat{\Pi}_j$ which minimizes the average probability of error P_e. Applying the quantum Bayes strategy, we obtain the necessary and sufficient condition for the POM $\hat{\Pi}_j$ to minimize the average probability of error P_e in the following form (Helstrom, 1969, 1976; Holevo, 1975; Yuen *et al.*, 1975):

$$\hat{\Pi}_j[p_j\hat{\rho}_j - p_k\hat{\rho}_k]\hat{\Pi}_k = 0 \tag{2.8}$$

$$\hat{\Gamma} - p_j\rho_j \geq 0, \tag{2.9}$$

where $\hat{\Gamma}$ is the Lagrange operator defined by

$$\hat{\Gamma} = \sum_{j=1}^{M} p_j\hat{\rho}_j\hat{\Pi}_j \tag{2.10}$$

which becomes Hermitian from the first condition given by (2.8). When the POM $\hat{\Pi}_j$ satisfies (2.8) and (2.9), the minimum value of the average probability of error is expressed as

$$\begin{aligned} P_{\min} &\equiv \min_{\{\hat{\Pi}_j\}} P_e \\ &= 1 - \mathrm{Tr}\,\hat{\Gamma} \end{aligned} \tag{2.11}$$

In our case, we set $p_j = 1/M$ and $\hat{\rho}_j = \hat{V}^{j-1}|\psi\rangle\langle\psi|\hat{V}^{\dagger j-1}$ in equations (2.8)–(2.10).

It is very difficult to obtain the POM which minimizes the average probability of error by solving equations (2.8)–(2.10). However, we can obtain the analytical solution for the quantum states which satisfy equations (2.1)–(2.3). Now we will prove the following proposition.

Proposition 1. When the quantum states satisfy the conditions given by equations (2.1)–(2.3), the optimum measurement that minimizes the average probability of error P_e is described by the POM $\hat{\Pi}_j$ which is defined by

$$\hat{\Pi}_j = |\mu_j\rangle\langle\mu_j| \tag{2.12}$$

$$|\mu_j\rangle = \hat{\Phi}^{-1/2}|\psi_j\rangle \tag{2.13}$$

$$\hat{\Phi} = \sum_{j=1}^{M} |\psi_j\rangle\langle\psi_j| \tag{2.14}$$

where the state vector $|\mu_j\rangle$ is called the optimum measurement state. In this case, the minimum value of the average probability of error is given by

$$P_{\min} = 1 - |\langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle|^2 \tag{2.15}$$

The minimum value of the average probability of error and the optimum POM for the symmetric quantum states given by equations (2.1)–(2.3) have been obtained by means of the eigenstate-expansion method (Helstrom, 1976). But the results are too complicated to be applied in quantum measurement theory and quantum information/communication theory. On the other hand, the optimum POM given by equations (2.12)–(2.14) is useful for applications (Holevo, 1979; Hausladen and Wootters, 1994; Hausladen *et al.*, 1996).

Proof of the Proposition. To prove the proposition, we will show that the POM $\hat{\Pi}_j$ given by (2.12)–(2.14) satisfies the necessary and sufficient conditions given by (2.8)–(2.10). From the definition of the operator $\hat{\Phi}$ and the relation $\hat{V}^M = \hat{I}$, we obtain

$$\begin{aligned}
\hat{V}\hat{\Phi}\hat{V}^\dagger &= \sum_{j=1}^{M} \hat{V}|\psi_j\rangle\langle\psi_j|\hat{V}^\dagger \\
&= \sum_{j=2}^{M} |\psi_j\rangle\langle\psi_j| + |\psi_{M+1}\rangle\langle\psi_{M+1}| \\
&= \sum_{j=2}^{M} |\psi_j\rangle\langle\psi_j| + |\psi_1\rangle\langle\psi_1| \\
&= \hat{\Phi}
\end{aligned} \tag{2.16}$$

Since the operator $\hat{V}$ is unitary, the nonnegative Hermitian operator $\hat{\Phi}$ commutes with the unitary operator $\hat{V}$, that is,

$$[\hat{\Phi}, \hat{V}] = 0 \tag{2.17}$$

Thus the operators $\hat{\Phi}$ and $\hat{V}$ can be simultaneously diagonalized and expressed in the following forms:

$$\hat{\Phi} = \sum_{\lambda\in S} \gamma_\lambda |\phi_\lambda\rangle\langle\phi_\lambda| \tag{2.18}$$

$$\hat{V} = \sum_{\lambda\in S} e^{-i\theta_\lambda} |\phi_\lambda\rangle\langle\phi_\lambda| \tag{2.19}$$

where S is the index set characterizing the spectrum of the operator. In equation (2.19), we have used the unitality of the operator $\hat{V}$. Since $\hat{\Phi}$ is a nonnegative Hermitian operator with $\mathrm{Tr}\,\hat{\Phi} = M$, the eigenvalue γ_λ satisfies $\gamma_\lambda \geq 0$ and $\sum_{\lambda\in S} \gamma_\lambda = M$. Furthermore, the relation $\hat{V}^M = \hat{I}$ requires that the parameter θ_λ should be $\theta_\lambda = 2\pi n_\lambda/M$ with integer n_λ. Moreover, since the operator $\hat{\Phi}$ is Hermitian, the set of the eigenstates $\{|\phi_\lambda\rangle | \lambda \in S\}$ spans a complete orthonormal system in a Hilbert space $\mathcal{H}$,

$$\langle\phi_\lambda|\phi_{\lambda'}\rangle = \delta_{\lambda,\lambda'} \tag{2.20}$$

$$\sum_{\lambda\in S} |\phi_\lambda\rangle\langle\phi_\lambda| = \hat{I} \tag{2.21}$$

In the following, we will use the relation $[\hat{\Phi}^{\pm 1/2}, \hat{V}^j] = 0$. Now we show that the POM $\hat{\Pi}_j$ given by equations (2.12)–(2.14) satisfies the first condition given by equation (2.8). Substituting equations (2.1), (2.3), and (2.12)–(2.14) into equation (2.8), we find for the left-hand side of equation (2.8)

$$\hat{\Pi}_j[p_j\hat{\rho}_j - p_k\hat{\rho}_k]\hat{\Pi}_k = \frac{1}{M}|\mu_j\rangle\langle\mu_j|(|\psi_j\rangle\langle\psi_j| - |\psi_k\rangle\langle\psi_k|)|\mu_k\rangle\langle\mu_k|$$
$$\equiv \frac{1}{M}|\mu_j\rangle\mathcal{F}_{jk}\langle\mu_k| \tag{2.22}$$

Using equations (2.2) and (2.17), we can calculate the quantity $\mathcal{F}_{jk}$ as follows:

$$\begin{aligned}\mathcal{F}_{jk} &= \langle\mu_j|\psi_j\rangle\langle\psi_j|\mu_k\rangle - \langle\mu_j|\psi_k\rangle\langle\psi_k|\mu_k\rangle \\ &= \langle\psi|\hat{V}^{\dagger j-1}\hat{\Phi}^{-1/2}\hat{V}^{j-1}|\psi\rangle\langle\psi|\hat{V}^{\dagger j-1}\hat{\Phi}^{-1/2}\hat{V}^{k-1}|\psi\rangle \\ &\quad - \langle\psi|\hat{V}^{\dagger j-1}\hat{\Phi}^{-1/2}\hat{V}^{k-1}|\psi\rangle\langle\psi|\hat{V}^{\dagger k-1}\hat{\Phi}^{-1/2}\hat{V}^{k-1}|\psi\rangle \\ &= \langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle\langle\psi|\hat{\Phi}^{-1/2}\hat{V}^{k-j}|\psi\rangle - \langle\psi|\hat{\Phi}^{-1/2}\hat{V}^{k-j}|\psi\rangle\langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle \\ &= 0\end{aligned} \tag{2.23}$$

Therefore we have found that the first condition given by equation (2.8) is satisfied by the POM $\hat{\Pi}_j$ given by equations (2.12)–(2.14). To consider the second condition, we first calculate the Lagrange operator $\hat{\Gamma}$ given by equation (2.10). Substituting equations (2.1), (2.3), and (2.12)–(2.14) into (2.10), we obtain

$$\begin{aligned}\hat{\Gamma} &= \frac{1}{M}\sum_{j=1}^{M}|\psi_j\rangle\langle\psi_j|\mu_j\rangle\langle\mu_j| \\ &= \frac{1}{M}\sum_{j=1}^{M}|\psi_j\rangle\langle\psi|\hat{V}^{\dagger j-1}\hat{\Phi}^{-1/2}\hat{V}^{j-1}|\psi\rangle\langle\mu_j| \\ &= \frac{1}{M}\langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle\sum_{j=1}^{M}|\psi_j\rangle\langle\mu_j| \\ &= \frac{1}{M}\langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle\sum_{j=1}^{M}|\psi_j\rangle\langle\psi_j|\hat{\Phi}^{-1/2} \\ &= \frac{1}{M}\langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle\hat{\Phi}\hat{\Phi}^{-1/2} \\ &= \frac{1}{M}\langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle\hat{\Phi}^{1/2}\end{aligned} \tag{2.24}$$

Thus, for our purpose, we have to show that the following Hermitian operator is nonnegative:

$$\begin{aligned}
\langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle\hat{\Phi}^{1/2} &- |\psi_j\rangle\langle\psi_j| \\
&= \hat{V}^{j-1}[\langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle\hat{\Phi}^{1/2} - |\psi\rangle\langle\psi|]\hat{V}^{\dagger j-1} \\
&\equiv \hat{V}^{j-1}\hat{\mathscr{G}}\hat{V}^{\dagger j-1}
\end{aligned} \tag{2.25}$$

Since the unitary transform of the nonnegative Hermitian operator is also nonnegative Hermitian, it is enough to show the inequality $\hat{\mathscr{G}} \geq 0$, which means that $\langle u|\hat{\mathscr{G}}|u\rangle \geq 0$ for any vector $|u\rangle$ in the Hilbert space $\mathscr{H}$. Using the Schwarz inequality such that $\langle A|A\rangle\langle B|B\rangle \geq |\langle A|B\rangle|^2$, we can calculate as follows:

$$\begin{aligned}
\langle u|\hat{\mathscr{G}}|u\rangle &= \langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle\langle u|\hat{\Phi}^{1/2}|u\rangle - \langle u|\psi\rangle\langle\psi|u\rangle \\
&= \langle\psi|\hat{\Phi}^{-1/4}\hat{\Phi}^{-1/4}|\psi\rangle\langle u|\hat{\Phi}^{1/4}\hat{\Phi}^{1/4}|u\rangle - |\langle\psi|u\rangle|^2 \\
&\geq |\langle\psi|\hat{\Phi}^{-1/4}\hat{\Phi}^{1/4}|u\rangle|^2 - |\langle\psi|u\rangle|^2 \\
&= |\langle\psi|u\rangle|^2 - |\langle\psi|u\rangle|^2 \\
&= 0
\end{aligned} \tag{2.26}$$

This result indicates that the second condition given by equation (2.9) is fulfilled. Therefore we have shown that the POM $\hat{\Pi}_j$ given by equations (2.12)–(2.14) minimizes the average probability of error P_e. ■

Using the above result, we can express the minimum value of the average probability of error in the following form:

$$\begin{aligned}
P_{\min} &= 1 - |\langle\psi|\hat{\Phi}^{-1/2}|\psi\rangle|^2 \\
&= 1 - \left(\sum_{\lambda\in S} \gamma_\lambda^{-1/2}|\langle\phi_\lambda|\psi\rangle|^2\right)^2
\end{aligned} \tag{2.27}$$

The optimum POM $\hat{\Pi}_j$ which minimizes the average probability of error is expressed as

$$\hat{\Pi}_j = |\mu_j\rangle\langle\mu_j| \tag{2.28}$$

$$|\mu_j\rangle = \sum_{\lambda\in S} \gamma_\lambda^{-1/2} e^{-i\theta_\lambda(j-1)}\langle\phi_\lambda|\psi\rangle|\phi_\lambda\rangle \tag{2.29}$$

Furthermore, the conditional probability $P(j|k)$ is given by

$$\begin{aligned}
P(j|k) &= |\langle\mu_j|\psi_k\rangle|^2 \\
&= |(\mathscr{G}^{1/2})_{jk}|^2
\end{aligned} \tag{2.30}$$

where $\mathscr{G}$ is the Gram matrix of the quantum states, that is, $\mathscr{G}_{jk} = \langle\psi_j|\psi_k\rangle$. These results are valid for both linearly independent and linearly dependent quantum states when equations (2.1)–(2.3) are satisfied. The optimum POM $\hat{\Pi}_j$ given by equations (2.12)–(2.14) has been used as the decoding observable by Hausladen *et al.* (1996) to investigate the classical information transmitted through a quantum channel. However, the optimality of the decoding observable was not mentioned there. If the state vectors $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_M\rangle$ are orthogonal, it is clear that the optimum POM given by equations (2.12)–(2.14) becomes the projection operator $\hat{\Pi}_j = |\psi_j\rangle\langle\psi_j|$.

2.2. Extremum of the Mutual Information

Thus far we have considered the minimization of the average probability of error. Here we will investigate the maximization of the mutual information obtained in the measurement for the symmetric quantum states defined by equations (2.1)–(2.3). The mutual information I is given by (Cover and Thomas, 1991)

$$I = \sum_{j=1}^{M} \sum_{k=1}^{M} p_k P(j|k) \ln\left[\frac{P(j|k)}{\sum_{n=1}^{M} P(j|n)p_n}\right] \tag{2.31}$$

where the conditional probability $P(j|k)$ is defined by equation (2.6). In equation (2.31), the information is measured in nats. If the quantum measurement described by the POM $\hat{\Pi}_j$ maximizes the mutual information, the following relation is satisfied (Holevo, 1973):

$$\hat{\Pi}_j[\hat{F}_j - \hat{F}_k]\hat{\Pi}_k = 0 \tag{2.32}$$

where the Hermitian operator $\hat{F}_j$ is defined by

$$\hat{F}_j = \sum_{k=1}^{M} p_k \hat{\rho}_k \ln\left[\frac{P(j|k)}{\sum_{n=1}^{M} P(j|n)p_n}\right] \tag{2.33}$$

This is the necessary, but not sufficient, condition for the POM $\hat{\Pi}_j$ to maximize the mutual information I. To our knowledge, the sufficient condition has not been found. The condition given by equation (2.32) is equivalent to $\delta I = 0$, where δI is the first variation with respect to the POM $\hat{\Pi}_j$ (Holevo, 1973), and the condition equivalent to $\delta^2 I < 0$ is not known. For the maximization of the mutual information, we have the following proposition.

Proposition 2. When the quantum states are specified by equations (2.1)–(2.3), the POM $\hat{\Pi}_j$ given by equations (2.12)–(2.14) satisfies the necessary condition (2.32) for the maximization of the mutual information.

Proof of the Proposition. To prove the proposition, it should be noted that the conditional probability $P(j|k)$ given by equation (2.6) can be expressed as

$$P(j|k) = |\langle\psi|\hat{\Phi}^{-1/2}\hat{V}^{j-k}|\psi\rangle|^2 \equiv f(j-k) \tag{2.34}$$

which satisfies the relations

$$f(j) = f(-j) \tag{2.35}$$

$$f(j \pm M) = f(j) \tag{2.36}$$

It is found from these relations that $\Sigma_{k=1}^{M} P(j|k)p_k = (1/M)\,\Sigma_{j=1}^{M} P(j|k) = 1/M$. Then the operator $\hat{F}_j - \hat{F}_k$ is calculated as

$$\begin{aligned}
\hat{F}_j - \hat{F}_k &= \frac{1}{M}\sum_{n=1}^{M} \hat{\rho}_n[\ln f(j-n) - \ln f(k-n)] \\
&= \frac{1}{M}\sum_{n=1}^{M} [\hat{\rho}_{n+j-1} - \hat{\rho}_{n-k-1}]\ln f(n) \\
&= \frac{1}{M}\sum_{n=1}^{M} [\hat{\rho}_{n+j-1} - \hat{\rho}_{k-n+1}]\ln f(n)
\end{aligned} \tag{2.37}$$

where we have used the relation $\hat{\rho}_{n\pm M} = \hat{\rho}_n$ and equations (2.35) and (2.36). Thus the left-hand side of equation (2.32) becomes

$$\begin{aligned}
\hat{\Pi}_j[\hat{F}_j - \hat{F}_k]\,\hat{\Pi}_k &= \frac{1}{M}\sum_{n=1}^{M} \hat{\Pi}_j[\hat{\rho}_{n+j-1} - \hat{\rho}_{k-n+1}]\,\hat{\Pi}_k \ln f(n) \\
&\equiv \frac{1}{M}|\mu_j\rangle\left(\sum_{n=1}^{M} \mathcal{F}_{jk}(n)\ln f(n)\right)\langle\mu_k|
\end{aligned} \tag{2.38}$$

and the quantity $\mathcal{F}_{jk}(n)$ is calculated as

$$\begin{aligned}
\mathcal{F}_{jk}(n) &= \langle\mu_j|\psi_{n+j-1}\rangle\langle\psi_{n+j-1}|\mu_k\rangle - \langle\mu_j|\psi_{k-n+1}\rangle\langle\psi_{k-n+1}|\mu_k\rangle \\
&= \langle\psi|\hat{\Phi}^{-1/2}\hat{V}^{n-1}|\psi\rangle\langle\psi|\hat{\Phi}^{-1/2}\hat{V}^{k-j-n+1}|\psi\rangle \\
&\quad - \langle\psi|\hat{\Phi}^{-1/2}\hat{V}^{k-j-n+1}|\psi\rangle\langle\psi|\hat{\Phi}^{-1/2}\hat{V}^{n-1}|\psi\rangle \\
&= 0
\end{aligned} \tag{2.39}$$

Therefore the POM $\hat{\Pi}_j$ given by equations (2.12)–(2.14) satisfies the necessary condition (2.32) for the maximization of the mutual information. ■

In the quantum measurement described by the optimum POM Π_j given by equations (2.12)–(2.14), we can obtain the information I_M on the M symmetric quantum states,

$$I_M = \ln M + \sum_{j=1}^{M} f(j)\ln f(j) \tag{2.40}$$

where $f(j)$ is given by equation (2.34) and we have used equations (2.35) and (2.36). It should be noted that $f(j)$ satisfies the relations

$$f(0) = f(M) = 1 - P_{\text{min}} \tag{2.41}$$

$$\sum_{j=n+1}^{M+n} f(j) = 1 \tag{2.42}$$

where P_{min} is the minimum value of the average probability of error given by equation (2.27) and n is an arbitrary integer. It is clear that the first term on the right-hand side of equation (2.40) represents the maximum value of the classical information that the physical system can carry and the second term represents the loss of the information due to the detection error. It is important to note that equation (2.40) is extremum, but not necessarily maximum. For $M = 1, 2$, we can obtain

$$I_2 = \ln 2 - H_{\text{bin}}(P_{\text{min}}) \tag{2.43}$$

$$I_3 = \ln 3 - P_{\text{min}} \ln 2 - H_{\text{bin}}(P_{\text{min}}) \tag{2.44}$$

where $H_{\text{bin}}(p) = -p \ln p - (1 - p)\ln(1 - p)$ is the binary entropic function (Cover and Thomas, 1991).

Hausladen and Wootters (1994) discussed that the mutual information becomes nearly maximum in the quantum measurement described by the POM $\hat{\Pi}_j$ given by (2.12)–(2.14). They calculated the first variation δI and showed $\delta I \approx 0$. The proposition we have just proved makes their result more rigorous for the symmetric quantum states. From Propositions 1 and 2, the POM $\hat{\Pi}_j$ given by equations (2.12)–(2.14) minimizes the average probability of error P_e and satisfies the necessary condition for the maximization of the mutual information I when the quantum states are pure and symmetric and have equal *a priori* probabilities.

2.3. Simple Examples

It is instructive to consider simple examples of the symmetric quantum states given by equations (2.1)–(2.3). Let us obtain the minimum value of the average probability of error and the optimum measurement states $|\mu_j\rangle$ for two coherent states $|\psi_1\rangle = |\alpha\rangle$ and $|\psi_2\rangle = |\beta\rangle$. This problem has been already solved by several methods (Helstrom, 1967, 1968; Osaki *et al.*, 1996a; Ban *et al.*, 1996). In this case, we have

$$\hat{V} = \hat{D}(\alpha + \beta)(-1)^{\hat{a}^\dagger \hat{a}} \tag{2.45}$$

$$|\psi\rangle = |\alpha\rangle \tag{2.46}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are bosonic annihilation and creation operators, respectively, satisfying the commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{I}$, and $\hat{D}(\alpha)$ is the displacement operator, namely $\hat{D}(\alpha) = \exp[\alpha\hat{a}^\dagger - \alpha^*\hat{a}]$. Using the relation

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha + \beta)e^{i\mathrm{Im}(\alpha\beta^*)}$$

it is easy to verify that

$$\begin{aligned}\hat{\rho}_j &= |\psi_j\rangle\langle\psi_j| \\ &= \hat{V}^{j-1}|\alpha\rangle\langle\alpha|\hat{V}^{\dagger j-1}\end{aligned} \tag{2.47}$$

with $j = 1, 2$. To solve the problem, we have to obtain the simultaneous eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ of the operators $\hat{\Phi} = |\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|$ and $\hat{V}$. It is an easy task to obtain

$$|\phi_1\rangle = \frac{|\alpha\rangle + e^{-i\varphi}|\beta\rangle}{\sqrt{2(1 + \kappa)}} \tag{2.48}$$

$$|\phi_2\rangle = \frac{|\alpha\rangle - e^{-i\varphi}|\beta\rangle}{\sqrt{2(1 - \kappa)}} \tag{2.49}$$

which satisfy $\langle\phi_j|\phi_k\rangle = \delta_{jk}$, where the parameters κ and φ are given by

$$\kappa = \exp\left[-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \mathrm{Re}(\alpha^*\beta)\right] \tag{2.50}$$

$$\varphi = \mathrm{Im}(\alpha^*\beta) \tag{2.51}$$

Then the operators $\hat{\Phi}$ and $\hat{V}$ are expressed as

$$\hat{\Phi} = \sum_{k=1,2} |\phi_k\rangle\gamma_k\langle\phi_k| \tag{2.52}$$

$$\hat{V} = \sum_{k=1,2} |\phi_k\rangle e^{-i\theta_k}\langle\phi_k| \tag{2.53}$$

with

$$\begin{cases}\gamma_1 = 1 + \kappa \\ \gamma_2 = 1 - \kappa\end{cases} \qquad \begin{cases}\theta_1 = 0 \\ \theta_2 = \pi\end{cases} \tag{2.54}$$

Thus the minimum value of the average probability of error becomes

$$\begin{aligned}P_{\min} &= 1 - \left(\sum_{k=1,2} \gamma_k^{-1/2}|\langle\alpha|\phi_k\rangle|^2\right)^2 \\ &= \frac{1}{2}(1 - \sqrt{1 - \kappa^2})\end{aligned} \tag{2.55}$$

and the optimum measurement states $|\mu_1\rangle$ and $|\mu_2\rangle$ are obtained from equation (2.29),

$$
\begin{aligned}
|\mu_1\rangle &= \sum_{k=1,2} \gamma_k^{-1/2}\langle\phi_k|\alpha\rangle|\phi_k\rangle \\
&= \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle) \\
&= \left(\frac{1+\sqrt{1-\kappa^2}}{1-\kappa^2}\right)^{1/2}|\alpha\rangle - e^{-i\varphi}\left(\frac{1-\sqrt{1-\kappa^2}}{1-\kappa^2}\right)^{1/2}|\beta\rangle
\end{aligned} \tag{2.56}
$$

$$
\begin{aligned}
|\mu_2\rangle &= \sum_{k=1,2} \gamma_k^{-1/2}e^{-i\theta_k}\langle\phi_k|\alpha\rangle|\phi_k\rangle \\
&= \frac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_2\rangle) \\
&= \left(\frac{1-\sqrt{1-\kappa^2}}{1-\kappa^2}\right)^{1/2}|\alpha\rangle - e^{i\varphi}\left(\frac{1+\sqrt{1-\kappa^2}}{1-\kappa^2}\right)^{1/2}|\beta\rangle
\end{aligned} \tag{2.57}
$$

Of course, these results are identical with those obtained in Helstrom (1967, 1968), Osaki *et al.* (1996a), and Ban *et al.* (1996). But the calculation is much easier in the present method. It has recently been found that the mutual information given by equation (2.43) becomes maximum in this optimum measurement (Osaki *et al.*, 1996b).

Next we consider an arbitrary binary quantum-state signal whose quantum states are given by $|\psi_1\rangle$ and $|\psi_2\rangle$ with $\langle\psi_1|\psi_2\rangle = \kappa e^{i\varphi}$. We define the unitary operator V as

$$\hat{V} = |\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2| \tag{2.58}$$

where the state vectors $|\phi_1\rangle$ and $|\phi_2\rangle$ are given by

$$|\phi_1\rangle = \frac{|\psi_1\rangle + e^{-i\varphi}|\psi_2\rangle}{\sqrt{2(1-\kappa)}} \tag{2.59}$$

$$|\phi_2\rangle = \frac{|\psi_1\rangle - e^{-i\varphi}|\psi_2\rangle}{\sqrt{2(1-\kappa)}} \tag{2.60}$$

Then we can easily check the following relations:

$$\hat{V}^\dagger\hat{V} = \hat{V}\hat{V}^\dagger = \hat{V}^2 = \hat{I} \tag{2.61}$$

$$\hat{\rho}_j = |\psi_j\rangle\langle\psi_j| = \hat{V}^{j-1}|\psi_1\rangle\langle\psi_1|\hat{V}^{\dagger j-1} \tag{2.62}$$

where $\hat{I}$ is an identity operator defined on the Hilbert space spanned by the state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$,

$$\hat{I} = \frac{|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| - |\psi_1\rangle\langle\psi_1|\psi_2\rangle\langle\psi_2| - |\psi_2\rangle\langle\psi_2|\psi_1\rangle\langle\psi_1|}{1 - |\langle\psi_1|\psi_2\rangle|^2} \tag{2.63}$$

Therefore, Propositions 1 and 2 are valid for an arbitrary binary quantum-state signal with equal *a priori* probabilities.

Finally let us consider linearly dependent spin-1/2 quantum states defined by $|\psi_j\rangle = \hat{V}^{j-1}|\psi\rangle$ and $|\psi\rangle = \binom{1}{0}$, where the unitary operator $\hat{V}$ is given by

$$\hat{V} = \begin{pmatrix} \cos[2\pi/M] & -\sin[2\pi/M] \\ \sin[2\pi/M] & \cos[2\pi/M] \end{pmatrix} \tag{2.64}$$

It is easy to see that the relation $\sum_{j=1}^{M} |\psi_j\rangle = 0$ is satisfied. In this case, the nonnegative Hermitian operator $\hat{\Phi}$ becomes diagonal,

$$\begin{aligned} \hat{\Phi} &= \sum_{j=1}^{M} |\psi_j\rangle\langle\phi_j| \\ &= \frac{1}{2} M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{2.65}$$

Thus it is easy to see that the optimum measurement states is $|\mu_j\rangle = \sqrt{(2/M)}|\psi_j\rangle$ and the minimum value of the average probability of error is $P_{\min} = 1 - (2/M)$, which are equal to those obtained in Helstrom (1976).

3. PARAMETER ESTIMATION FOR SYMMETRIC QUANTUM STATE

3.1. Maximum-Likelihood Estimation

In the preceding section, we have shown that the quantum measurement described by the POM Π_j given by equations (2.12)–(2.14) minimizes the average probability of error and makes the mutual information extremum for the symmetric quantum states defined by equations (2.1)–(2.3). In this section, we will prove that the continuous version of equations (2.12)–(2.14) becomes optimum for the maximum-likelihood estimation for the parameters of symmetric quantum states. The quantum state whose parameter is estimated from the result of the measurement is given by

$$\begin{aligned} \hat{\rho}(\theta) &= |\psi(\theta)\rangle\langle\psi(\theta)| \\ &= \hat{V}(\theta)|\psi\rangle\langle\psi|\hat{V}^{\dagger}(\theta) \end{aligned} \tag{3.1}$$

where θ is the parameter to be estimated and the operator $\hat{V}(\theta)$ satisfies the relations

$$\hat{V}(\theta)\hat{V}^{\dagger}(\theta) = \hat{V}^{\dagger}(\theta)\hat{V}(\theta) = \hat{I} \tag{3.2}$$

$$\hat{V}(\theta_1)\hat{V}(\theta_2) = \hat{V}(\theta_1 + \theta_2) \tag{3.3}$$

$$\hat{V}(\theta + l/2) = \hat{V}(\theta - l/2) \tag{3.4}$$

with $\hat{V}(0) = \hat{I}$. Thus the operator $\hat{V}(\theta)$ is unitary and periodic with respect to the parameter θ. Thus we can restrict the range of the parameter θ to be $-\frac{1}{2}l \leq \theta < \frac{1}{2}l$. Furthermore, we assume that there is not any prior knowledge of values of the parameter θ. Thus *a priori* probability $p(\theta)$ of the value of the parameter θ becomes

$$p(\theta) = 1/l \tag{3.5}$$

The quantum measurement whose result indicates the value θ of the parameter is described by the POM $\hat{\Pi}(\theta)$ (Helstrom, 1976; Holevo, 1982), which satisfies the relations

$$\hat{\Pi}(\theta) \geq 0 \tag{3.6}$$

$$\int_{-l/2}^{l/2} d\theta\, \hat{\Pi}(\theta) = \hat{I} \tag{3.7}$$

We have to obtain the optimum POM $\hat{\Pi}(\theta)$ such that we can estimate the value of the parameter as accurately as possible.

When we apply the maximum-likelihood estimation (Helstrom, 1976; Milburn *et al.*, 1994), we obtain the necessary and sufficient conditions for the POM $\hat{\Pi}(\theta)$ to be optimum,

$$[\hat{\Upsilon} - p(\theta)\hat{\rho}(\theta)]\hat{\Pi}(\theta) = 0 \tag{3.8}$$

$$\hat{\Upsilon} - p(\theta)\hat{\rho}(\theta) \geq 0 \tag{3.9}$$

where $\hat{\Upsilon}$ is the Lagrange operator defined by

$$\hat{\Upsilon} = \int_{-l/2}^{l/2} d\theta\, p(\theta)\hat{\rho}(\theta)\hat{\Pi}(\theta) \tag{3.10}$$

which is required to be Hermitian. In our case, we set $\hat{\rho}(\theta) = \hat{V}(\theta)|\psi\rangle\langle\psi|\hat{V}^{\dagger}(\theta)$ and $p(\theta) = 1/l$ in equations (3.8)–(3.10). Then we obtain the following proposition.

Proposition 3. When the quantum state satisfies the conditions given by (3.1)–(3.5), the optimum measurement by which we can estimate, as accurately as possible, the value of the parameter of the quantum state is described by the POM $\hat{\Pi}(\theta)$ defined by

$$\hat{\Pi}(\theta) = |\mu(\theta)\rangle\langle\mu(\theta)| \tag{3.11}$$

$$|\mu(\theta)\rangle = \hat{\Psi}^{-1/2}|\psi(\theta)\rangle \tag{3.12}$$

$$\hat{\Psi} = \int_{-l/2}^{l/2} d\theta\ |\psi(\theta)\rangle\langle\psi(\theta)| \tag{3.13}$$

Proof of the Proposition. To prove the proposition, we will show that the POM $\hat{\Pi}(\theta)$ given by (3.11)–(3.13) satisfies the necessary and sufficient conditions given by (3.8) and (3.9). For our purpose, it should be noted that the following relation is obtained from (3.2)–(3.4):

$$\begin{aligned}
\hat{V}(\theta)\hat{\Psi}\hat{V}^{\dagger}(\theta) &= \int_{-l/2}^{l/2} d\phi\ \hat{V}(\theta)|\psi(\phi)\rangle\langle\psi(\phi)|\hat{V}^{\dagger}(\theta) \\
&= \int_{-l/2}^{l/2} d\phi\ |\psi(\phi + \theta)\rangle\langle\psi(\phi + \theta)| \\
&= \int_{-l/2}^{l/2} d\phi\ |\psi(\phi)\rangle\langle\psi(\phi)| \\
&= \hat{\Psi}
\end{aligned} \tag{3.14}$$

where we have used the periodicity of the quantum state with respect to the parameter. Since the operator $\hat{V}(\theta)$ is unitary, the nonnegative Hermitian operator $\hat{\Psi}$ commutes with the unitary operator $\hat{V}(\theta)$, that is,

$$[\hat{\Psi}, \hat{V}(\theta)] = 0 \tag{3.15}$$

Using this result, we calculate the Lagrange operator $\hat{\Upsilon}$ given by (3.10) as follows:

$$\begin{aligned}
\hat{\Upsilon} &= \frac{1}{l}\int_{-l/2}^{l/2} d\theta\ |\psi(\theta)\rangle\langle\psi(\theta)|\mu(\theta)\rangle\langle\mu(\theta)| \\
&= \frac{1}{l}\int_{-l/2}^{l/2} d\theta\ |\psi(\theta)\rangle\langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle\langle\psi(\theta)|\hat{\Psi}^{-1/2} \\
&= \frac{1}{l}\langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle \int_{-l/2}^{l/2} d\theta\ |\psi(\theta)\rangle\langle\psi(\theta)|\hat{\Psi}^{-1/2} \\
&= \frac{1}{2}\langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle\hat{\Psi}\hat{\Psi}^{-1/2} \\
&= \frac{1}{l}\langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle\hat{\Psi}^{1/2}
\end{aligned} \tag{3.16}$$

This indicates that the Lagrange operator $\hat{\Psi}$ is Hermitian. Let us now show that the POM $\hat{\Pi}(\theta)$ given by equations (3.11)–(3.13) satisfies the first condition given by (3.8). Substituting (3.1), (3.11)–(3.13), and (3.16) into (3.8), we obtain

$$\begin{aligned}
&\left[\hat{\Upsilon} - \frac{1}{l}\,\hat{\rho}(\theta)\right]\hat{\Pi}(\theta)\\
&= \frac{1}{l}\,[\langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle\hat{\Psi}^{1/2} - |\psi(\theta)\rangle\langle\psi(\theta)|]\hat{\Psi}^{-1/2}|\psi(\theta)\rangle\langle\psi(\theta)|\hat{\Psi}^{-1/2}\\
&= \frac{1}{l}\,\langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle|\psi(\theta)\rangle\langle\psi(\theta)|\hat{\Psi}^{-1/2}\\
&\quad - \frac{1}{l}\,|\psi(\theta)\rangle\langle\psi(\theta)|\hat{\Psi}^{-1/2}|\psi(\theta)\rangle\langle\psi(\theta)|\hat{\Psi}^{-1/2}\\
&= \frac{1}{l}\,\langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle|\psi(\theta)\rangle\langle\psi(\theta)|\hat{\Psi}^{-1/2}\\
&\quad - \frac{1}{l}\,|\psi(\theta)\rangle\langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle\langle\psi(\theta)|\hat{\Psi}^{-1/2}\\
&= 0.
\end{aligned} \tag{3.17}$$

Thus the first condition given by (3.8) is fulfilled. Next we will prove that the following Hermitian operator is nonnegative:

$$\begin{aligned}
\hat{\Upsilon} - p(\theta)\hat{\rho}(\theta) &= \frac{1}{l}\,[\langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle\hat{\Psi}^{1/2} - |\psi(\theta)\rangle\langle\psi(\theta)|]\\
&= \frac{1}{l}\,\hat{V}(\theta)[\langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle\hat{\Psi}^{1/2} - |\psi\rangle\langle\psi|]\hat{V}^{\dagger}(\theta)\\
&\equiv \frac{1}{l}\,\hat{V}(\theta)\hat{\mathcal{M}}\hat{V}^{\dagger}(\theta)
\end{aligned} \tag{3.18}$$

where we have used the commutation relation (3.15). Since the unitary transform of the nonnegative Hermitian operator is also nonnegative Hermitian, it is enough to show that the operator $\hat{\mathcal{M}}$ is nonnegative. Using the Schwarz inequality, we obtain for any vector $|u\rangle$ in the Hilbert space

$$\begin{aligned}
\langle u|\hat{\mathcal{M}}|u\rangle &= \langle\psi|\hat{\Psi}^{-1/2}|\psi\rangle\langle u|\hat{\Psi}^{1/2}|u\rangle - |\langle u|\psi\rangle|^2\\
&= \langle\psi|\hat{\Psi}^{-1/4}\hat{\Psi}^{-1/4}|\psi\rangle\langle u|\hat{\Psi}^{1/4}\hat{\Psi}^{1/4}|\psi\rangle - |\langle u|\psi\rangle|^2\\
&\geq |\langle u|\hat{\Psi}^{1/4}\hat{\Psi}^{-1/4}|\psi\rangle - |\langle u|\psi\rangle|^2\\
&= |\langle u|\psi\rangle|^2 - |\langle u|\psi\rangle|^2\\
&= 0
\end{aligned} \tag{3.19}$$

This result indicates that $\hat{\mathcal{M}} \geq 0$ and the second condition given by (3.9) is fulfilled. Therefore we have found that the POM $\hat{\Pi}(\theta)$ given by (3.11)–(3.13) satisfies the necessary and sufficient conditions in the maximum-likelihood estimation. ■

When the quantum measurement is described by the POM $\hat{\Pi}(\theta)$, the operator $\hat{\Theta}$ which represents the measured quantity is expressed as

$$\hat{\Theta} = \int d\theta \cdot \theta \hat{\Pi}(\theta) \tag{3.20}$$

In general, since the POM $\hat{\Pi}(\theta)$ is not a projection operator, the operator $\hat{\Theta}$ defined by (3.20) does not become Hermitian. So, the operator $\hat{\Theta}$ is sometimes called the semiobservable. In the quantum measurement, the conditional probability $P(\phi \mid \theta)$ that the estimated value of the parameter is ϕ when the value θ is true is expressed as

$$\begin{aligned} P(\phi \mid \theta) &= P(\phi - \theta) \\ &= \mathrm{Tr}[\hat{\Pi}(\phi)\hat{\rho}(\theta)] \\ &= |\langle \psi | \hat{\Psi}^{-1/2} | \psi(\phi - \theta)\rangle|^2 \\ &= |\mathcal{G}^{1/2}(\phi - \theta)|^2 \end{aligned} \tag{3.21}$$

where $\mathcal{G}(\phi - \theta) = \mathcal{G}(\phi, \theta) = \langle \psi(\phi) | \psi(\theta)\rangle$ is the continuous version of the Gram matrix of the quantum states.

3.2. Simple Examples

Before closing this section, we will consider two simple examples, the phase estimation and position estimation. The optimum measurements have already been obtained by solving (3.8)–(3.10). Here we directly calculate (3.11)–(3.13), which is much easier. First we obtain the optimum measurement for phase estimation of the quantum state. In this case, the quantum state is given by

$$|\psi(\theta)\rangle = \exp[-i\theta \hat{a}^\dagger \hat{a}] |\psi\rangle \tag{3.22}$$

with $-\pi \leq \theta < \pi$. Then we have $l = 2\pi$. When we expand the quantum state $|\psi\rangle$ as $|\psi\rangle = \Sigma_{n=0}^{\infty} \psi_n |n\rangle$, the operator $\hat{\Psi}$ is calculated to be

$$\hat{\Psi} = 2\pi \sum_{n=0}^{\infty} |\psi_n|^2 |n\rangle\langle n| \tag{3.23}$$

which is diagonal in the Fock-state basis. Thus the optimum POM is easily obtained as

$$\hat{\Pi}(\theta) = |\mu(\theta)\rangle\langle\mu(\theta)| \tag{3.24}$$

$$|\mu(\theta)\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{\psi_n}{|\psi_n|} e^{-i\theta n} |n\rangle \tag{3.25}$$

where we have assumed $\psi_n \neq 0$ for simplicity. In particular, when the parameter ψ_n is positive for all n, the quantum state $|\mu(\theta)\rangle$ becomes the Susskind–Glogower phase eigenstate and the semiobservable in this optimum measurement is the Susskind–Glogower phase operator $\hat{\phi}_{SG}$ (Susskind and Glogower, 1964; Carruthers and Nieto, 1968), which is not Hermitian,

$$\hat{\phi}_{SG} = \int_{-\pi}^{\pi} d\phi \, |\phi\rangle\phi\langle\phi| \tag{3.26}$$

$$\exp[-i\hat{\phi}_{SG}] = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \tag{3.27}$$

with

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \exp(-in\phi)|n\rangle \tag{3.28}$$

This means that the best strategy for the phase estimation is to measure the Susskind–Glogower phase operator $\hat{\psi}_{SG}$.

Next let us consider a one-dimensional physical system which extends between $x = -\frac{1}{2}l$ and $x = \frac{1}{2}l$. We assume the periodic boundary condition $|\psi(\frac{1}{2}l)\rangle = |\psi(-\frac{1}{2}l)\rangle$. In this case, the state vector $|\psi(x)\rangle$ can be expressed as

$$|\psi(x)\rangle = \exp[-i\hat{p}x]|\psi\rangle \tag{3.29}$$

where $\hat{p}$ is the momentum operator of the system whose spectrum is given by $p_n = 2\pi n/l$ with $n = 0, \pm 1, \pm 2, \ldots, \pm\infty$. Then the operator $\hat{\Psi}$ is easily calculated,

$$\hat{\Psi} = \int_{-l/2}^{l/2} dx \, |\psi(x)\rangle\langle\psi(x)| = l \sum_{n=-\infty}^{\infty} |\psi_n|^2 |p_n\rangle\langle p_n| \tag{3.30}$$

where $|p_n\rangle$ is the eigenstate of the momentum operator with the eigenvalue $p_n = 2\pi n/l$ and $\psi_n = \langle p_n|\psi\rangle$. Thus the optimum POM $\hat{\Pi}(x)$ is given by

$$\hat{\Pi}(x) = |\mu(x)\rangle\langle\mu(x)| \tag{3.31}$$

$$|\mu(x)\rangle = \frac{1}{\sqrt{l}} \sum_{n=-\infty}^{\infty} \frac{\psi_n}{|\psi_n|} e^{-ip_n x}|p_n\rangle \tag{3.32}$$

When the parameter ψ_n is positive for all n, the state vector $|\mu(x)\rangle$ becomes the eigenstate of the position operator. In the limit of $l \to \infty$, the best strategy for the position-parameter estimation of the quantum state is to measure the position observable $\hat{x}$, which is Hermitian. Although the results given by equations (3.25) and (3.32) were obtained by solving equations (3.8)–(3.10), the calculation of equations (3.11)–(3.13) is much easier.

4. SUMMARY

We have proved that in the sense of the minimization of the average probability of error, the POM given by equations (2.12)–(2.14) describes the optimum quantum measurement for discrimination among the symmetric quantum states with equal *a priori* probabilities. We have also shown that the POM satisfies the necessary condition for the maximization of the mutual information. Thus the mutual information obtained in the measurement is extremum. Furthermore, applying the maximum-likelihood estimation, we have found that the POM given by equations (3.11)–(3.13) describes the optimum quantum measurement for the parameter estimation of the symmetric quantum state. The optimum POM for the parameter estimation has the same structure as that of the discrimination of the quantum states. We have given simple examples to illustrate our results.

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